Last time: - integration

- briefly, L^p spaces

Today: More L^p spaces, L^2 space

Fix a measure space $(X, \mathcal{M}, \mu)$.
We say that a property of points in X holds for almost every $x \in X$, or holds almost everywhere, if there exists a $N \in \mathcal{M}$ such that $\mu(N) = 0$ and the property holds for all points in $X \setminus N$.

E.g. If a measurable function f is 0 almost everywhere, then
$$\int f \, d\mu = 0.$$

Lem: A measurable function f is 0 a.e. iff
$$\int |f| \, d\mu = 0.$$

Def: Let $p \in [1, \infty)$.
Let $f : X \to \mathbb{C}$ be measurable.
The **p-norm** of f is
$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{1/p}.$$ (could be ∞)

We also the **∞-norm** to be
$$\|f\|_\infty = \inf \{ c \geq 0 \mid |f(x)| \leq c \text{ a.e.} \}.$$
$(= \infty$ if this set $\}$ is empty$)$

If $\lambda \in \mathbb{C}$ then $\|\lambda f\|_p = |\lambda| \, \|f\|_p$ (Homogeneity).

Lemma (Minkowski's inequality):
If $f, g : X \to \mathbb{C}$ are measurable and $p \in [1, \infty]$

then $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

The proof will use

Lemma (Hölder's inequality):

If $f,g : X \to \mathbb{C}$ are measurable and $p,q \in [1,\infty]$ satisfy $\frac{1}{p}+\frac{1}{q}=1$, (so $q = \frac{1}{1-\frac{1}{p}}$)

then $\|fg\|_1 \le \|f\|_p \|g\|_q$.

Pf: If $\|f\|_p = 0$ then $|f|^p = 0$ a.e.

so $|f| = 0$ a.e.

so $fg = 0$ a.e.

so $\|fg\|_1 = 0$,

in which case we are done.

So we may assume $\|f\|_p > 0$,

and similarly that $\|g\|_q > 0$.

If either is $+\infty$ then the RHS is $+\infty$

so we are done.

So we may assume that $\|f\|_p, \|g\|_q \in (0,\infty)$.

Define $\tilde{f} = \dfrac{f}{\|f\|_p}$, $\tilde{g} = \dfrac{g}{\|g\|_q}$.

Note we just need to show

$$\|\tilde{f}\tilde{g}\|_1 \leq 1.$$

Recall from the warm-up Young's inequality:

if $a,b \geq 0$ then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

Thus
$$\int |\tilde{f}\tilde{g}|\, d\mu \leq \int \left(\frac{|\tilde{f}|^p}{p} + \frac{|\tilde{g}|^q}{q} \right) d\mu$$
$$= \frac{1}{p}\int |\tilde{f}|^p d\mu + \frac{1}{q}\int |\tilde{g}|^q d\mu$$
$$= \frac{1}{p}\|\tilde{f}\|_p^p + \frac{1}{q}\|\tilde{g}\|_q^q$$

Now we note that $\|\tilde{f}\|_p = \left\| \frac{f}{\|f\|_p} \right\|_p$

$$= \frac{1}{\|f\|_p} \|f\|_p$$

$$= 1$$

Similarly $\|g\|_q = 1$.

Since $\frac{1}{p} + \frac{1}{q} = 1$, we are done.

$\square$

Later to prove $L^p(X)$ is Banach, we will want

Thm (**Monotone Convergence**):

If (f_n) is a sequence of measurable functions $X \to [0,\infty]$ converging to a function f, and $f_n(x) \le f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in X$, then f is measurable and

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

PF: Since $f \geq f_n$ for all n,

$$\int f \, d\mu \geq \int f_n \, d\mu \quad \text{for all } n$$

$$\Rightarrow \int f \, d\mu \geq \lim_n \int f_n \, d\mu .$$

Since $f = \liminf_n f_n$

$$\int f \, d\mu \leq \liminf_n \int f_n \, d\mu \quad \text{(Fatou)}$$

$$= \lim_n \int f_n \, d\mu .$$

$\square$

PF of Minkowski inequality:

$$\|f+g\|_p^p = \int |f+g|^p \, d\mu$$

$$= \int |f+g| \cdot |f+g|^{p-1} \, d\mu$$

$$\leq \int (|f| + |g|) \, |f+g|^{p-1} \, d\mu$$

$$= \int |f| \, |f+g|^{p-1} \, d\mu + \int |g| \, |f+g|^{p-1} \, d\mu$$

$$= \| f \, |f+g|^{p-1} \|_1 + \| g \, |f+g|^{p-1} \|_1$$

[Hölder] $$\leq \|f\|_p \, \|(f+g)^{p-1}\|_q + \|g\|_p \, \|(f+g)^{p-1}\|_q$$

$$= (\|f\|_p + \|g\|_p) \, \|(f+g)^{p-1}\|_q$$

$$= (\|f\|_p + \|g\|_p) \left(\int |f+g|^{(p-1)q} d\mu\right)^{1/q}$$

$$= (\|f\|_p + \|g\|_p) \left(\int |f+g|^p d\mu\right)^{1-\frac{1}{p}}$$

$$= (\|f\|_p + \|g\|_p) \frac{\|f+g\|_p^p}{\|f+g\|_p}$$

$$\Rightarrow \|f+g\|_p \|f+g\|_p^p \leq (\|f\|_p + \|g\|_p) \|f+g\|_p^p$$

(*)

If $\|f+g\|_p^p = 0$, then we are done

If $\|f\|_p = \infty$ or $\|g\|_p = \infty$ then we are done.

Assume both are finite.

Then we can bound

$$\left|\frac{f}{2} + \frac{g}{2}\right|^p \leq \frac{|f|^p}{2} + \frac{|g|^p}{2}$$

Using Jensen's inequality from the warm-up for $f(x) = x^p$

So $|f+g|^p \leq 2^{p-1}(|f|^p + |g|^p)$

$\Rightarrow \quad \|f+g\|_p^p \leq 2^{p-1}(\|f\|_p^p + \|g\|_p^p) < \infty$

Thus we can cancel $\|f+g\|_p^p$ term in $(*)$ to get $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

$\square$

We now have shown that the p-norm is a function on the vector space of measurable functions which satisfies Homogeneity and the Triangle Inequality. Such a function is called a seminorm, In this case we can turn it into a norm by changing our space of interest.

Def: The $\mathcal{L}^p$-space or pre-L^p-space on X is defined for $p \in [1,\infty]$ by

$$\mathcal{L}^p(X) = \{ f : X \to \mathbb{C} \text{ measurable} \mid \|f\|_p < \infty \}.$$

The L^p-space on X is

$$L^p(X) = \mathcal{L}^p(X) / \{ f \in \mathcal{L}^p(X) \mid f = 0 \text{ a.e.} \}$$

Note that $f = 0$ a.e. iff $\|f\|_p = 0$.

Taking this quotient makes it so that $\|f\|_p = 0$ iff $f = 0$ in $L^p(X)$, as required by a norm.

Thus, $L^p(X)$ is a normed space for all $p \in [1,\infty]$.

In fact it is Banach.

To prove this, we use all of our integral theorems so far.

$$L^2(\mathbb{N}) = \ell^2(\mathbb{N}) = \{ f : \mathbb{N} \to \mathbb{C} \mid \sum_{n \in \mathbb{N}} |f(n)|^2 < \infty \}$$

Thm (Riesz-Fischer Thm)

$L^p(X)$ is a Banach space.

Pf: we use

Lem: A normed space is Banach iff every absolutely convergent series is convergent.

Pf: $\Rightarrow$) warm-up

$\Leftarrow$ turn a subsequence of a Cauchy sequence into a telescoping series which absolutely converges.

$\square$

Let (f_n) be a sequence of L^p functions such that

$$\sum_n \|f_n\|_p < \infty.$$

Now note $\left(\sum_{n=1}^{N} |f_n|\right)^p$ is an increasing sequence of measurable fns as $N \to \infty$.

So by the monotone convergence thm

$$\int \left(\sum_{n=1}^{\infty} |f_n| \right)^p d\mu = \lim_{N \to \infty} \int \left(\sum_{n=1}^{N} |f_n| \right)^p d\mu$$

$$= \lim_{N \to \infty} \left\| \sum_{n=1}^{N} |f_n| \right\|_p^p$$

(Minkowski) $$\leq \lim_{N \to \infty} \left(\sum_{n=1}^{N} \|f_n\|_p \right)^p$$

$$= \left(\sum_{n=1}^{\infty} \|f_n\|_p \right)^p < \infty .$$

In particular, $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ a.e., so $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise a.e.

So if f is the (a.e.) pointwise sum $\sum_{n=1}^{\infty} f_n$,

this shows that $f \in L^p$

since $\|f\|_p^p \leq \int \left(\sum_{n=1}^{\infty} |f_n| \right)^p d\mu$.

Need to show that the partial sums converge to f in the p-norm.

I.e. need to show

$$\lim_{N\to\infty} \| f - \underbrace{\sum_{n=1}^{N} f_n}_{g_N} \|_p = 0$$

$\forall N \in \mathbb{N}$

$$|g_N|^p \leq \left(\sum_{n=1}^{\infty} |f_n| \right)^p,$$ which is integrable

So by dominated convergence,

$$\lim_{N\to\infty} \int |g_N|^p \, d\mu = \int \lim_{N\to\infty} |g_N|^p \, d\mu$$

$$= 0.$$

Thus $L^p(X)$ is a Banach space. $\square$